THE KING'S SCHOOL, CANTERBURY

SCHOLARSHIP ENTRANCE EXAMINATION

March 2011

MATHEMATICS 2

Time: 1 hour (plus reading time)

Use the reading time wisely; gain an overview of the paper and start to think of how you will answer the questions.

Do as many questions as you can and in any order you like (clearly numbered) on the lined paper provided.

The questions are not of equal length or mark allocation. Make sure you avoid spending too much time on any one question; don't get bogged down! Move on quickly if you get stuck. The paper is quite long; you are not necessarily expected to finish everything.

Some of the later questions are more difficult, but not necessarily longer. Some questions are designed to test your ability to work with unfamiliar ideas, or familiar ones with a twist. Don't give up!

You are expected to use a calculator where appropriate, but also you must show ***full and clear working****, diagrams and arguments wherever you can. Marks will be awarded for method as well as answers. In fact, merely writing down an answer might score few marks.*

Complete questions are preferable to fragments. You can sometimes, however, manage to complete later parts of questions, even if you have failed to answer the earlier sections.

This paper has eight questions.

1 In this question all dates are given in the eight-digit format **DD.MM.YYYY** (so today's date is 01.03.2011)

Recently there was a **palindromic full date** of 11.02.2011, where the digits read the same forwards and backwards.

(Note that, for instance, 1st October 2010 does not count as palindromic in this question, since we write it here as 01.10.2011 and not 1.10.2011)

(a) When will the next palindromic full date occur?

(b) How many other palindromic full dates (i.e. after your answer to (a)) will there be before the start of the year 2030?

(c) Work out the nearest palindromic full date that occurred **before** 10.02.2001.

2 Here are the recent (20 February 2011) standings in the English Football Premier League. Notice how the only four teams whose names begin with the letter W occupy the bottom four places.

1 Man Utd
2 Arsenal
3 Man City
4 Tottenham
5 Chelsea
6 Liverpool
7 Sunderland
8 Bolton
9 Newcastle
10 Stoke
11 Blackburn
12 Fulham
13 Everton
14 Birmingham
15 Aston Villa
16 Blackpool
17 West Brom
18 Wigan
19 West Ham
20 Wolves

Answer the questions on the facing page.

(a) Explain why the number of ways of filling the bottom four places (where the order matters) is

$$20 \times 19 \times 18 \times 17 = 116\,280$$

(b) How many ways are there if we are not bothered about the order of the teams anymore, just which four they are?

[So, for instance, we would consider a league table finishing West Brom, Wigan, West Ham, Wolves the same as the otherwise identical one finishing Wigan, Wolves, West Ham, West Brom (etc etc).]

(c) If I wrote the complete list of 20 teams down in random order, what would the probability be that I put the teams with names beginning with W in the bottom four (in any order)?

(d) Comment on the size of your answer.

3 *This question is a fourth-century problem from a collection called the Greek Anthology, and concerns the mythological tale of the fifth of the twelve tasks of Heracles:* ***Cleaning the Stables of Augeas.***

Heracles the mighty was questioning Augeas, seeking to know the **number of his herds**, and Augeas replied:

“About the streams of Alpheius, my friend, are **half** of them; the **eighth** part pasture around the hill of Cronos, the **twelfth** part far away by the precinct of Taraxippus; the **twentieth** part feed in holy Elis, and I left the **thirtieth** part in Arcadia; but here you see the **remaining fifty** herds.”

The question you must answer is: how many herds did Augeas have?

[Hint: call this number *x*, use the information in the paragraph above to form an equation in *x* and solve it.]

4 In this question you can assume the truth of the **Midpoint Theorem**, namely:

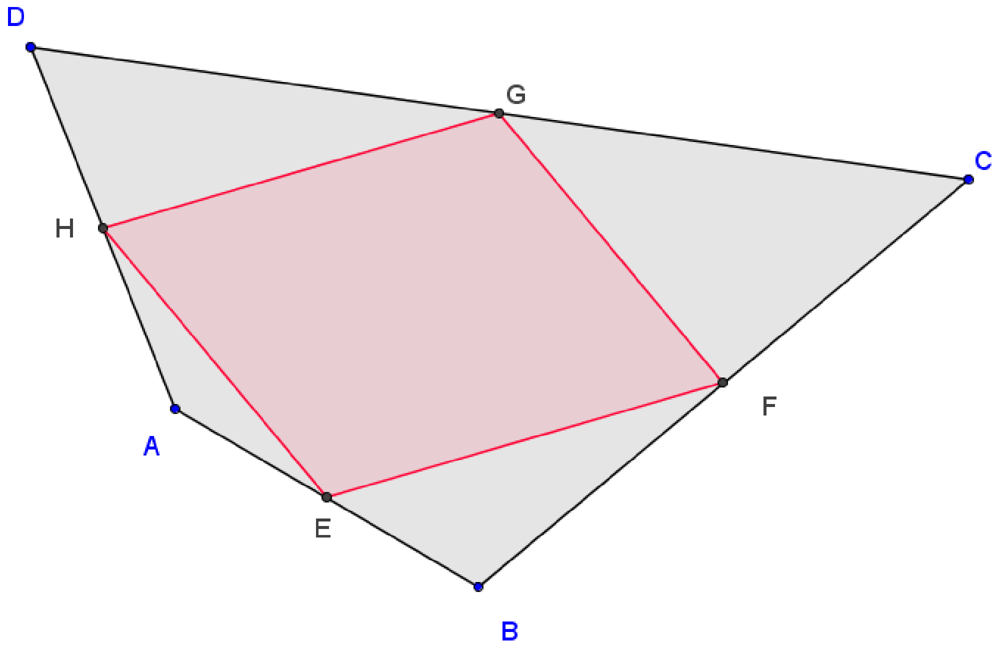
The line which joins the midpoints of two sides of a triangle is parallel to the third side and is equal to half of the length of the third side.

(a) Draw and label a diagram to show this result.

In this question we are going to prove **Varignon's Theorem**, which states:

The figure formed when the midpoints of the sides of a convex quadrilateral are joined in order is a parallelogram.

[Convex means: do not worry about quadrilaterals shaped like an arrowhead, or whose sides cross over]



(b) Use the Midpoint Theorem to prove that Varignon's Theorem is correct. Write down your argument carefully, using the letters in the diagram to denote the appropriate angles and sides.

[Hint: redraw the diagram above and add in the diagonals of the original quadrilateral]

(c) Can you now draw a new quadrilateral for which the Varignon "parallelogram" we draw would actually be a **square**?

Write a sentence underneath to say what you did. What is the most general quadrilateral you can draw which has this property?

What is special about the original quadrilateral you have drawn (I do not mean the square)?

(d) Can you guess a general result about the relationship between the area of the original quadrilateral and that of the Varignon parallelogram?

Can you explain why your guess is always true?

5 [In this question you must show your working/reasoning carefully for each part.]

You meet some current pupils at King’s while sitting the Scholarship examination.

In this question **Shells** always tell the truth and **Removes** always lie. In parts (a) to (c) everyone is either a Shell or a Remove.

(a) Consider two pupils, Andrew and Ben. Andrew says “at least one of us is a Remove”. What are Andrew and Ben?

(b) Now forget about part (a). Cath says “either I am a Remove or Deborah is a Shell”. What are Cath and Deborah?

(c) Now consider three pupils: Ellie, Fred, Gail.

Ellie says “all of us are Removes”

Fred says “exactly one of us is a Shell”.

What are Ellie, Fred and Gail?

Now, you can also meet another type of pupil called a **Fifth**, who sometimes lies and sometimes tells the truth.

Nick says “I am a Fifth”.

Dave says “I agree with Nick; that is true”.

Ed says “I am not a Fifth”.

(d) Suppose you know that of Nick, Dave, Ed exactly one each is a Shell, Remove and Fifth. Which is which?

6 We are used to writing down numbers in **decimal** (base 10) notation, so we can write, for instance, the number three hundred and fifty six as 356, which actually represents

$$356 = 3\times10^2 + 5\times10^1 + 6\times1$$

If we decide to use a **binary** (base 2) notation instead then we will write a number as powers of 2 and what we write will only be ones and zeroes.

For example, the number we might normally write as 27 can be seen as

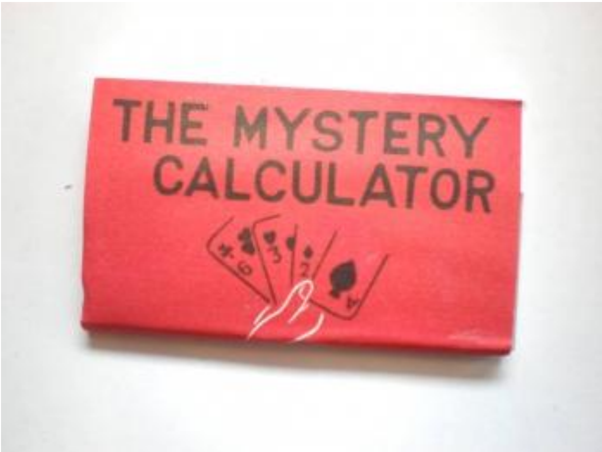
$$1\times2^4 + 0\times2^3 + 0\times2^2 + 1\times2^1 + 1\times1$$

so in binary we write this number as 10011.

(a) What is the binary number 110011 in the usual decimal notation?

(b) Write down the decimal number 53 in binary.

Last December at my House Christmas party I found a game in my Christmas cracker called Mystery Calculator, which contained six cards, shown on the facing page.



On the back were written the rules of the game:

The complete set consists of 6 cards printed with a series of numbers. Show all the cards to a friend and ask him of her to select one number from any one card. Show the other five cards to your friend asking him or her to say whether the number appears on these cards. Take all the cards on which your friend says the number appears, add together the top left hand corner number of each card and the total is the number your friend selected.

(c) Can you explain very carefully why this trick works? [Hint: think about the early parts of this question.]

1 3 5 7 9 11 13 15
17 19 21 23 25 27 29 31
33 35 37 39 41 43 45 47
49 51 53 55 57 59 61 63

2 3 6 7 10 11 14 15
18 19 22 23 26 27 30 31
34 35 38 39 42 43 46 47
50 51 54 55 58 59 62 63

4 5 6 7 12 13 14 15
20 21 22 23 28 29 30 31
36 37 38 39 44 45 46 47
52 53 54 55 60 61 62 63

8 9 10 11 12 13 14 15
24 25 26 27 28 29 30 31
40 41 42 43 44 45 46 47
56 57 58 59 60 61 62 63

16 17 18 19 20 21 22 23
24 25 26 27 28 29 30 31
48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63

32 33 34 35 36 37 38 39
40 41 42 43 44 45 46 47
48 49 50 51 52 53 54 55
56 57 58 59 60 61 62 63

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7 Solve this equation in *x*:

$$\frac{x}{1+\frac{1}{x}} - \frac{x}{1-\frac{1}{x}} = 1$$

[Hint: you might start by multiplying the numerators and denominators of the two left-hand side fractions by something appropriate.]

8 [Hint: in this question you might like to draw a Venn Diagram or similar]

Mr Miles wishes to introduce new items of uniform for scholars at King's, but he and the other teachers cannot decide on whether they would like all or some of a bag, a scarf or a special hat.

It is decided that the scholars will settle the matter with a vote. All scholars vote for or against each of the three.

One quarter vote for a bag. Two thirds vote for a scarf.

Ten vote for a bag but were against the other two.

Of those who were in favour of a bag, 14 cast only one other positive vote.

None of those who voted for a hat also voted for a scarf.

Of those who voted for just one of the options, 54 more voted for a scarf than for a hat.

No-one voted against all three.

(a) How many scholars are there at King's?

(b) How many voted for a hat?

(c) How many voted for both a bag and a hat?

END OF PAPER